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# An existence result for some semi-linear elliptic equation in bent strip-like unbounded domains (Variational Problems and Related Topics)

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# An existence result for some semi-linear elliptic equation in bent strip-like unbounded domains

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## 1 Introduction and Main Result

Let  $N \geq 2$  and  $\Omega$  be an unbounded domain in  $\mathbf{R}^N$ . We consider the following equation

$$\begin{cases} -\Delta u + \lambda u = u_+^p & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \quad (1)$$

where  $\lambda \geq 0$  and  $1 < p < \infty$  if  $N = 2$ ,  $1 < p < (N+2)/(N-2)$  if  $N \geq 3$  are given constants. It is well-known that (1) has a positive solution if  $\Omega$  is bounded. In general, the existence of a positive solution of (1) is unknown if  $\Omega$  is unbounded. Esteban and Lions showed in [4] that if  $\Omega$  satisfies following condition (EL) then there is no nontrivial solution.

(EL) There exists a vector  $X \in \mathbf{R}^N$  such that  $\nu(x) \cdot X \geq 0$  and  $\nu(x) \cdot X \not\equiv 0$  on  $x \in \partial\Omega$ , where  $\nu(x)$  is the outer unit normal vector of  $\Omega$ .

On the other hand, many authors showed existence result. (cf. [1, 3, 5, 7] and references therein). In this paper, we will give an existence result in bent strip-like unbounded domains. We use following notations.

$$\begin{aligned} S_d &:= \{x = (x', x_N) \in \mathbf{R}^N; |x'| < d\}, \\ \hat{S}_d &:= \{x = (x', x_N) \in S_d; x_N > 0\}. \end{aligned}$$

In [6], we conjectured that if  $\lambda \geq 0$  and  $\Omega$  satisfying the following condition ( $\Omega 1$ ) then there is a nontrivial solution.

( $\Omega 1$ )  $\Omega$  is a domain in  $\mathbf{R}^N$  and  $\partial\Omega$  is Lipschitz continuous. There are  $K \in \mathbf{N} \setminus \{1\}$ , a bounded set  $A$  and congruent transformations  $\Lambda_j$  ( $1 \leq j \leq K$ ) such that  $\Omega = A \cup \Lambda_1(\hat{S}_d) \cup \cdots \cup \Lambda_K(\hat{S}_d)$  and  $\Lambda_i(\hat{S}_d) \cap \Lambda_j(\hat{S}_d) = \emptyset$  if  $i \neq j$ .

This conjecture is still open. In this paper, we consider the following stronger conditions ( $\Omega 2$ ), ( $\Omega 3$ ) in two dimensional case. Here after, we assume  $N = 2$ .

(Ω2) There are  $d > 0$ , a smooth curve  $\{c(s)\}_{s \in \mathbb{R}}$  parameterized by arc length with the curvature  $\kappa(s)$  such that  $\text{supp}\{\kappa\}$  is compact and  $\Phi : S_d \rightarrow \Omega$  is bijective, where  $\Phi$  is defined by  $\Phi(y) := c(y_2) + y_1 e(y_2)$  and  $e(s)$  is the unit normal vector of  $c(s)$ .

(Ω3)  $\Omega$  satisfies (Ω1),  $\exists \Omega_0 \subset \Omega$  s.t.  $\Omega_0$  satisfies (Ω2).

**Remark.** If  $\Omega$  satisfies (Ω2) then

$$\Omega = \{x \in \mathbb{R}^2; \text{dist}(x, \{c(s)\}) < d\}.$$

So  $\Omega$  is a bent strip-like domain.

**Remark.**  $\Omega$  satisfies (Ω2) then  $\Omega$  satisfies (Ω3) with  $\Omega = \Omega_0$ .  $\Omega$  satisfies (Ω3) then  $\Omega$  satisfies (Ω1).

Now we state our main theorem.

**Theorem A.** Suppose that  $N = 2$ ,  $\lambda \geq 0$  and the following equation has unique nontrivial solution up to  $x_2$  transformation.

$$\begin{cases} -\Delta v + \lambda v = v_+^p & \text{in } S, \\ v \in H_0^1(S). \end{cases} \quad (2)$$

If  $(\|\kappa\|_{L^\infty} d)^2 < 1 - 2^{(1-p)/(1+p)}$  then (1) has a nontrivial solution.

**Remark.** If  $\lambda = 0$ , (2) has unique nontrivial solution up to  $x_2$  transformation by [2].

## 2 Preliminaries

At first, we state notations. For a domain  $D$ , we define following notations.

$$I[u] := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u_+^{p+1} dx \quad \text{for } u \in H_0^1(D) \subset H_0^1(\mathbb{R}^N),$$

$$M(D) := \{u \in H_0^1(D) \setminus \{0\}; \int_D |\nabla u|^2 + \lambda u^2 dx = \int_D u_+^{p+1}\},$$

$$\alpha(\Omega) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_D[\gamma(t)],$$

$$\Gamma := \{\gamma \in C([0,1]; H_0^1(D)); \gamma(0) = 0, I_D[\gamma(1)] \leq 0\}.$$

It is well-known that the mountain pass energy  $\alpha(D)$  is well-defined and is equal to a least energy. i.e.

**Lemma 2.1.** *Let  $D$  be a domain. Suppose that  $D$  satisfying Poincare's inequality or  $\lambda > 0$ . Then*

$$\alpha(D) = \inf_{u \in M(D)} I_D[u]$$

*and all nontrivial critical point  $v$  of  $I_D$  satisfies  $I_D[v] \geq \alpha(D)$ .*

(cf. [9]).

**Lemma 2.2.** *If  $\Omega$  satisfies  $(\Omega 1)$ . Then Poincare's inequality holds. i.e. There exists a constant  $C > 0$  such that*

$$\int_{\Omega} u^2 dx \leq C \int_{\Omega} |\nabla u|^2 dx.$$

By Lemma 2.2, we can use the norm

$$\|v\|_{H_0^1(\Omega)}^2 = \int_{\Omega} |\nabla v|^2 dx.$$

**Lemma 2.3.** *Let  $K$  be a complete metric space,  $K_0 \subset K$  be a closed set,  $X$  be a Banach space and  $\chi \in C(K_0, X)$ . Define  $\Gamma$  by*

$$\Gamma := \{\gamma \in C(K, X); \gamma(s) = \chi(s) \text{ if } s \in K_0\}.$$

*For  $I \in C^1(X, \mathbb{R})$ , put*

$$c := \inf_{\gamma \in \Gamma} \max_{s \in K} I[\gamma(s)], \quad c_1 := \max_{v \in K_0} I[\chi(v)].$$

*If  $c > c_1$  then for all  $\epsilon > 0$  and  $\gamma \in \Gamma$  with  $\max_{s \in K} I[\gamma(s)] \leq c + \epsilon$ , there exists  $v \in X$  such that*

$$c - \epsilon < I[v] < \max_{s \in K} I[\gamma(s)], \quad \text{dist}(v, g(K)) \leq \epsilon^{\frac{1}{2}}, \quad |I'[v]| \leq \epsilon^{\frac{1}{2}}.$$

*Especially, there is a Palais-Smale sequence.*

For the proof of this Lemma, see [8, Theorem 4.3].

**Proposition 2.4 (Concentration Compactness).** *Suppose  $(\Omega 1)$ . Let  $\{u_n\}_{n=1}^{\infty}$  be nonnegative Palais-Smale  $\beta$ -sequence for  $I_{\Omega}$  in  $H_0^1(\Omega)$ . i.e.*

$$I_{\Omega}[u_n] = \beta + o(1), \quad I'_{\Omega}[u_n] = o(1) \quad \text{as } n \rightarrow \infty.$$

Then there exist a non-negative number  $l$ ,  $k_1, \dots, k_l \in \{1, \dots, k\}$ ,  $\{z_n^i\}_{n=1}^\infty \subset \Lambda_{k_i}(\{x = (x', x_N); x' = 0\})$ ,  $u^0 \in H_0^1(\Omega)$  with  $u \geq 0$ ,  $u^i \in H_0^1(\Lambda_{k_i}(S))$  with  $u^i > 0$  for  $1 \leq i \leq l$  such that

$$\begin{aligned} u_n(x) &= u^0(x) + u^1(x - z_n^1) + \dots + u^l(x - z_n^l) + o(1) \\ &\text{as } n \rightarrow \infty \text{ in } H_0^1(\mathbf{R}^N), \\ I_\Omega[u_n] &= I_\Omega[u^0] + I_{\mathbf{R}^N}[u^1] + \dots + I_{\mathbf{R}^N}[u^l] + o(1) \text{ as } n \rightarrow \infty, \\ \begin{cases} -\Delta u^0 + \lambda u^0 = (u^0)^p & \text{in } \Omega, \\ -\Delta u^i + \lambda u^i = (u^i)^p & \text{in } \Lambda_{k_i}(S), \end{cases} \\ |z_n^i| &\rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

We can give the proof of Lemma 2.4 by using same argument as in [7]. For reader's convenience, we give the proof in Appendix. To prove theorem A, we use the following functional. Take  $\phi \in C(\mathbf{R}^N, [-1, 1])$  satisfying

$$\phi(x) = \begin{cases} 1 & x \in \Lambda_i(S_0), i:\text{odd}, \\ -1 & x \in \Lambda_i(S_0), i:\text{even}, \\ 0 & \text{otherwise.} \end{cases}$$

Define the functional  $h : L^2(\mathbf{R}^N) \setminus \{0\} \rightarrow [-1, 1]$  by

$$h[u] := \frac{1}{\|u\|_{L^2(\mathbf{R}^N)}^2} \int_{\mathbf{R}^N} \phi(x) |u(x)|^2 dx \quad \text{for } u \in L^2(\mathbf{R}^N) \setminus \{0\}.$$

$h$  is a continuous function in the following sense.

**Lemma 2.5.** *There is a constant  $C > 0$  such that*

$$|h[u+v] - h[u]| \leq \frac{C(\|u\|_{L^2(\mathbf{R}^N)} + \|v\|_{L^2(\mathbf{R}^N)})}{\|u\|_{L^2(\mathbf{R}^N)}^2} \|v\|_{L^2(\mathbf{R}^N)}$$

for all  $u, v \in L^2(\mathbf{R}^N)$  with  $u \neq 0$  and  $u+v \neq 0$ . Especially,  $|h[u+v] - h[u]| \leq C\|v\|_{L^2(\mathbf{R}^N)}/\|u\|_{L^2(\mathbf{R}^N)}$  if  $\|v\|_{L^2(\mathbf{R}^N)} < \|u\|_{L^2(\mathbf{R}^N)}$ .

We can show Lemma 2.5 by elementary calculus. We omit the proof of

### 3 Proof of Theorem A and Theorem B

To prove Theorem A, we consider the following mountain-pass value  $\alpha_0(\Omega)$ . Put

$$\begin{aligned} H &= \{u \in H_0^1(\Omega); h[u] = 0\} \cup \{0\}, \\ \alpha_0(\Omega) &:= \inf_{\gamma \in \Gamma_0} \sup_{t \in [0,1]} I[\gamma(t)], \\ \Gamma_0 &:= \{\gamma \in C([0,1], H); g(0) = 0, I[g(1)] \leq 0\}. \end{aligned}$$

Here, it is easy to see that  $H$  is a closed subspace of  $H_0^1(\Omega)$ . By the definition of  $\alpha_0(\Omega)$ ,  $\alpha(\Omega) \leq \alpha_0(\Omega)$  holds. It is well-known that  $0 < \alpha(\Omega) \leq \alpha(S_d)$  if  $\Omega$  satisfies  $(\Omega 1)$  because of  $\alpha(\hat{S}_d) = \alpha(S_d)$ . So one of following cases holds.

- (a)  $\alpha(\Omega) < \alpha(S_d)$ .
- (b)  $\alpha(\Omega) = \alpha(S_d)$  and  $\alpha_0(\Omega) = \alpha(S_d)$ .
- (c)  $\alpha(\Omega) = \alpha(S_d)$  and  $\alpha_0(\Omega) > \alpha(S_d)$ .

**Proposition 3.1.** *Suppose that  $(\Omega 1)$ . If the case (a) or (b) holds then (1) has a positive solution.*

Proposition 3.1 is proved by standard arguments by using concentration compactness principle. We omit the proof of it. By Proposition 3.1, it is enough to show that Theorem A in the case (c). Hereafter, we suppose (c) and  $N = 2$ . For the proof of Theorem A, the least energy solution on  $S_d$  plays important role. Let  $v \in H_0^1(S_d)$  be a least energy solution on  $S_d$ . i.e.

$$\begin{cases} -\Delta v + \lambda v = v_+^p & \text{in } S_d, \\ v > 0 & \text{on } \partial S_d, \end{cases} \quad I[v] = \alpha(S_d).$$

The existence of such solution is well-known. By the moving plain method, we can assume that

$$v(x) = v(x_1, x_2) = v(|x_1|, |x_2|) \quad \text{for all } x \in S_d.$$

By the equation, we see

$$\int_{S_d} |\nabla v|^2 + \lambda v^2 dx = \int_{S_d} v_+^{p+1} dx. \quad (3)$$

Since  $(\Omega 3)$ ,  $\Psi := \Phi^{-1}$  is well-defined. Define  $v_t, u_t$  by

$$v_t(x) := v(\Psi_1(x), \Psi_2(x) - t), \quad u_t(x) = s(t)v_t(x),$$

where  $s(t)$  is uniquely determined positive constant satisfying  $u_t(x) \in M(\Omega)$  for each  $t$ . (see Lemma 4.1.)

**Lemma 3.2.** *If  $(d\|\kappa\|_{L^\infty(\mathbf{R})})^2 < 1 - 2^{(1-p)/(1+p)}$  then there exist constants  $t_0, s_0 > 0$  such that*

$$I[u_{\pm t_0}] < \frac{1}{2}(\alpha(S) + \alpha_0(\Omega)), \quad (4)$$

$$h[u_{t_0}] > \frac{1}{2}, \quad h[u_{-t_0}] < -\frac{1}{2}, \quad (5)$$

$$I[sv_t] \leq 0 \quad \text{if } s \geq s_0, \quad (6)$$

$$I[u_t] < 2\alpha(S) \quad \text{for all } t \in \mathbf{R}. \quad (7)$$

*Proof.* By elementally calculation for  $\Phi$ ,

$$\begin{aligned} I[sv_t] = & \frac{s^2}{2} \int_{S_d} \frac{1}{1 - y_1 \kappa(y_2)} v_{y_2}^2(y_1, y_2 - t) + (1 - y_1 \kappa(y_2)) v_{y_1}^2(y_1, y_2 - t) \\ & + \lambda(1 - y_1 \kappa(y_2)) v^2(y_1, y_2 - t) dy \\ & - \int_{S_d} (1 - y_1 \kappa(y_2)) F(sv(y_1, y_2 - t)) dy. \end{aligned}$$

Since  $v$  is even function with respect to  $y_1$  and  $1/(1+t) + 1/(1-t) = 2/(1-t^2)$ , we have

$$I[sv_t] = \frac{s^2}{2} \int_{S_d} \frac{1}{1 - (y_1 \kappa(y_2 + t))^2} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2 dy - \frac{1}{p+1} \int_{S_d} (sv)_+^{p+1} dy.$$

Since  $\frac{d}{ds} I[sv_t]|_{s=s(t)} = 0$ , we obtain

$$\int_{S_d} \frac{1}{1 - (y_1 \kappa(y_2 + t))^2} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2 dy = s(t)^{p-1} \int_{S_d} v_+^{p+1} dy \quad (8)$$

Here, the right hand side is increasing with respect to  $s$  and

$$\int_{S_d} \frac{1}{1 - (y_1 \kappa(y_2 + t))^2} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2 dy > \int_{S_d} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2 dy = \int_{S_d} v^{p+1} dy \quad (9)$$

by (3). So we have

$$s(t) \geq 1. \quad (10)$$

By using Lesbergue's convergence theorem, the left hand side of (9) tends to  $\int_{S_d} |\nabla v|^2 + \lambda v^2 dy$  as  $t \rightarrow \pm\infty$ . It and (3) mean  $s(t) \rightarrow 1$  as  $t \rightarrow \pm\infty$ . It asserts  $I[u_t] \rightarrow \alpha(S)$  as  $t \rightarrow \pm\infty$ . So (4) holds for sufficiently large  $t_0$ .

(8) and (3) assert

$$s(t)^{p-1} \leq \frac{1}{1 - (d\|\kappa\|_{L^\infty(\mathbf{R})})^2}. \quad (11)$$

By (8), (11) and the assumption of Theorem A, we can obtain

$$\begin{aligned} I[u_t] &= \left(\frac{1}{2} - \frac{1}{p+1}\right) s(t)^2 \int_{S_d} \frac{1}{1 - (y_1 \kappa(y_2 + t))^2} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2 dy \\ &\leq s(t)^2 \frac{1}{1 - (d\|\kappa\|_{L^\infty(\mathbf{R})})^2} \alpha(S_d) \\ &\leq \left(\frac{1}{1 - (d\|\kappa\|_{L^\infty(\mathbf{R})})^2}\right)^{\frac{p+1}{p-1}} \alpha(S_d) \\ &< 2\alpha(S_d). \end{aligned}$$

It means (7) holds for any  $t \in \mathbf{R}$ . It is easy to see that

$$I[sv_t] \leq \frac{s^2}{2} \frac{1}{1 - (d\|\kappa\|_{L^\infty(\mathbf{R})})^2} \int_{S_d} |\nabla v|^2 + \lambda v^2 dy - \frac{s^{p+1}}{p+1} \int_{S_d} v_+^{p+1} dy$$

The right hand side is independent of  $t$  and tends to  $-\infty$  as  $s \rightarrow \infty$ . So we obtain (6) for sufficiently large  $s_0$ .

By the assumption  $(\Omega 3)$  and the definition of  $v_t$ , we have

$$\|\chi_{\Lambda_1(S_d)} v_t - v_t\|_{L^2(\mathbf{R}^2)} \rightarrow 0 \text{ and } \|\chi_{\Lambda_1(S_d)} v_t\|_{L^2(\mathbf{R}^2)} \rightarrow \|v\|_{L^2(\mathbf{R}^2)} \neq 0.$$

Since  $h[\chi_{\Lambda_1(S_d)} v_t] = -1$  and Lemma 2.5, we obtain

$$h[v_t] \rightarrow -1 \quad \text{as } t \rightarrow -\infty.$$

Similarly,

$$h[v_t] \rightarrow 1 \quad \text{as } t \rightarrow \infty$$

holds. It completes the proof of Lemma 3.2.  $\square$

Put  $K := [0, s_0] \times [-t_0, t_0]$  and define  $\beta$  by

$$\beta := \inf_{\gamma \in \Gamma} \max_{(s,t) \in K} I[g(s,t)],$$

$$\Gamma_1 := \{\gamma \in C(S, H_0^1(\Omega)); g(s,t) = sv_t \text{ if } (s,t) \in \partial K\}.$$

Then the following Lemma 3.4 and Lemma 3.3 hold.



**Lemma 3.3.** *Suppose same assumptions as in Lemma 3.2 then*

$$\alpha(S) < \beta < 2\alpha(S).$$

**Lemma 3.4.** *Suppose same assumptions as in Lemma 3.2. Then there is a Palais-Smale  $\beta$ -sequence  $\{u_n\}_{n=1}^\infty$ . i.e.*

$$I[u_n] = \beta + o(1), \quad \|I[u_n]\| = o(1) \quad \text{as } n \rightarrow \infty.$$

*Proof of Lemma 3.3.* Put  $\gamma_0(s, t) = sv_t$  for  $(s, t) \in K$  then  $\gamma_0 \in \Gamma_1$ . By the assumption of  $f$ , we have  $I[sv_t] \leq I[u_t]$ . Lemma 3.2 asserts that  $I[\gamma_0(s, t)] \leq 2\alpha(S)$  for all  $(s, t) \in K$ . Hence  $\beta < 2\alpha(S)$ .

Fix any  $\gamma \in \Gamma_1$ , Lemma 3.2 and similar argument as in [10] show that there is a curve  $\tau : [0, 1] \rightarrow K$  such that  $\gamma \circ \tau \in \Gamma_0$ . So we have

$$\max_{(s,t) \in K} I[\gamma(s, t)] \geq \max_{t \in (0,1)} I[\gamma \circ \tau(t)] \geq \alpha_0(\Omega).$$

It means  $\alpha(S) < \beta$  by the condition (c). □

*Proof of Lemma 3.4.* Put  $\gamma_0(s, t) = sv_t$  for  $(s, t) \in K$  then  $\gamma_0 \in \Gamma_1$ , Lemma 3.2 asserts

$$\max_{(s,t) \in \partial K} I[\gamma_0(s, t)] \leq \frac{1}{2}(\alpha_0(\Omega) + \alpha(S)) < \beta.$$

So we can apply Lemma 2.3 to obtain the existence of Palais-Smale  $\beta$  sequence. □

Now we can prove Theorem B in the following Proposition.

**Proposition 3.5.** *Suppose that same assumption as in Theorem A. Then there is a positive solution.*

*Proof.* Let  $\{u_n\}_{n=1}^\infty$  be a Palais-Smale  $\beta$  sequence in Lemma 3.4. By Proposition 2.4, by passing to a subsequence if necessary, there is a nonnegative number  $l$  such that

$$\begin{aligned} u_n(x) &= u^0(x) + u^1(x - x_n^1) + \cdots + u^l(x - x_n^1) + o(1) & \text{as } n \rightarrow \infty \text{ in } H_0^1(\mathbb{R}^2), \\ I[u_n] &= I[u^0] + I[u^1] + \cdots + I[u^l] + o(1) & \text{as } n \rightarrow \infty. \end{aligned}$$

If  $u^0 \not\equiv 0$  then  $u^0$  is a positive solution. So it is enough to show that  $u^0 \not\equiv 0$ .

Suppose  $u \equiv 0$  then  $l \geq 1$  and

$$I[u_n] = I[u^1] + \cdots + I[u^l] + o(1) \geq l\alpha(S) + o(1) \quad \text{as } n \rightarrow \infty.$$

Since Lemma 3.4, we have  $\beta < 2\alpha(S)$ . So we can obtain  $l = 1$ . It mean that

$$I[u_n] = I[u^1] + o(1) \quad \text{as } n \rightarrow \infty.$$

Hence  $I[u_1] = \beta$ . So, we see that  $u_1(\Lambda_{k_1}(x))$  is a critical point of  $I$  in  $H_0^1(\Lambda_{k_1}(\hat{S}_d))$  with  $I[u_1(\Lambda_{k_1}(x))] = \beta$ . It contradicts to the uniqueness of nontrivial solutions on  $\Lambda_{k_1}(S_d)$ . Consequently, there exists a positive solution  $u^0$ .  $\square$

## 4 Appendix

In this section, we note well-known facts and give the proof of Proposition 2.4. First, we note some properties for  $f$ .

**Lemma 4.1.** *Suppose that  $D$  is a domain in  $\mathbf{R}^N$ . Fix  $v \in H_0^1(D)$  with  $v_+ \neq 0$  in  $H_0^1(D)$ . Then there is an uniquely determined constant  $s_0 > 0$  such that*

$$\left. \frac{d}{ds} I[sv] \right|_{s=s_0} = 0.$$

Moreover,

$$\max_{s>0} I[sv] = I[s_0 v].$$

*Proof.* We see

$$\frac{1}{s} \frac{d}{ds} I_D[sv] = \int_D |\nabla v|^2 + \lambda v^2 dx - s^{p-1} \int_D v_+^{p+1} dy$$

if  $s > 0$ . Second term of the right hand side is strictly decreasing with respect to  $s$  on  $(0, \infty)$ . Moreover, second term equals to 0 if  $s = 0$  and tends to  $-\infty$  as  $s \rightarrow \infty$ . Consequently, we obtain this Lemma.  $\square$

*Proof of Proposition 2.4.* By the assumption of  $u_n$ , we have

$$\begin{aligned} \langle I'[u_n], u_n \rangle &= \|u_n\|_{H_0^1(\Omega)}^2 + \lambda \|u_n\|_{L^2(\Omega)}^2 - \int_{\Omega} (u_n)_+^{p+1} dx = o(1) \|u_n\|_{H_0^1(\Omega)} \\ &\quad \text{as } n \rightarrow \infty. \end{aligned} \tag{12}$$

So we have

$$\begin{aligned} C \geq I[u_n] &= \left( \frac{1}{2} - \frac{1}{p+1} \right) (\|u_n\|_{H_0^1(\Omega)}^2 + \lambda \|u_n\|_{L^2(\Omega)}^2) + o(1) \|u_n\|_{H_0^1(\Omega)} \\ &\quad \text{as } n \rightarrow \infty. \end{aligned} \tag{13}$$

So we see that  $u_n$  is bounded in  $H_0^1(\Omega)$ . By using weak compactness for Hilbert space and Rellich's compactness, there exists  $u^0 \in H_0^1(\Omega)$  such that

$$\begin{aligned} u_n &\rightharpoonup u^0 && \text{weakly in } H_0^1(\Omega) \text{ as } n \rightarrow \infty, \\ u_n &\rightarrow u^0 && \text{in } L_{\text{loc}}^p(\Omega) \text{ as } n \rightarrow \infty, \\ u_n &\rightarrow u^0 && \text{a.e. in } \Omega \text{ as } n \rightarrow \infty, \end{aligned}$$

by passing to a subsequence if necessary. So we obtain

$$I'[u_n] \rightharpoonup I'[u^0] \quad \text{weakly in } H^{-1}(\Omega).$$

It means  $u^0$  is a critical point of  $I$ . Put  $\phi_n^1 := u_n - u^0$  then

$$\phi_n^1 \rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega) \text{ as } n \rightarrow \infty, \quad (14)$$

$$\phi_n^1 \rightarrow 0 \quad \text{in } L_{\text{loc}}^p(\Omega) \text{ as } n \rightarrow \infty. \quad (15)$$

Moreover, we have

$$\|\phi_n^1\|_{H_0^1(\Omega)}^2 = \|u_n\|_{H_0^1(\Omega)}^2 - \|u^0\|_{H_0^1(\Omega)}^2 + o(1) \quad \text{as } n \rightarrow \infty.$$

We can apply Brezis-Lieb's theorem to obtain

$$\int_{\Omega} (\phi_n^1)^{p+1} dx = \int_{\Omega} (u_n)^{p+1} dx - \int_{\Omega} (u^0)^{p+1} dx.$$

By using Vitali's Lemma, we have

$$I'[\phi_n^1] = I'[u_n] - I'[u^0] + o(1) = o(1) \quad \text{in } H^{-1}(\Omega) \text{ as } n \rightarrow \infty. \quad (16)$$

Suppose  $\phi_n^1 \rightarrow 0$  in  $H_0^1(\Omega)$  as  $n \rightarrow \infty$ , by passing to a subsequence if necessary. Then the proof is complete since  $u_n \rightarrow u^0$  in  $H_0^1(\Omega)$  as  $n \rightarrow \infty$ . So, hereafter, we can assume  $\phi_n^1$  is not convergence to 0 in  $H_0^1(\Omega)$  for any subsequence. Put

$$\begin{aligned} Q_0 &:= \Omega \setminus (\Lambda_1(\hat{S}_d) \cup \dots \cup \Lambda_k(\hat{S}_d)), \\ Q_m &:= \{x = (x', x_N) \in S; m-1 < x_N \leq m\}, \\ Q_m^j &:= \Lambda_j(Q_m) \end{aligned}$$

for  $m \geq 1, 1 \leq j \leq k$ . Define  $d_n$  and  $\hat{d}_n$  by

$$d_n := \max_{m \in \mathbb{N}, 1 \leq j \leq k} \|\phi_n^1\|_{L^2(Q_m^j)}, \quad \hat{d}_n := \max\{d_n, \|\phi_n^1\|_{L^2(Q_0)}\}$$

and show that

$$\liminf_{n \rightarrow \infty} \hat{d}_n > 0.$$

Since  $Q_n^j$  is congruence we can apply Sobolev's inequality to obtain

$$\|\phi_n^1\|_{L^r(Q_m^j)} \leq C(r) \|\phi_n^1\|_{H_0^1(Q_m^j)}$$

for  $q+1 < r \leq 2^*$  where  $C(q)$  is a positive constant independent of  $n, j$ . By using interpolation it holds that

$$\|\phi_n^1\|_{L^{q+1}(Q_m^j)}^{q+1} \leq C(r) \|\phi_n^1\|_{L^2(Q_m^j)}^{(1-\theta)(q+1)} \|\phi_n^1\|_{H_0^1(Q_m^j)}^{\theta(q+1)}$$

where  $1/(q+1) = (1-\theta)/2 + \theta/r$ . Since  $\theta \rightarrow 1$  as  $r \rightarrow q+1$ ,  $\theta(q+1) - 2 > 0$  for  $r$  near  $q+1$ . Fix such  $r$  then we have

$$\int_{Q_m^j} |\phi_n^1|^p dx \leq C d_n^{(1-\theta)(q+1)} \|\phi_n^1\|_{H_0^1(\Omega)}^{\theta(q+1)-2} \int_{Q_m^j} |\nabla \phi_n^1|^2 dx.$$

Similarly for  $Q_0$ , we have

$$\int_{Q_0} |\phi_n^1|^p dx \leq C \hat{d}_n^{(1-\theta)(q+1)} \|\phi_n^1\|_{H_0^1(\Omega)}^{\theta(q+1)-2} \int_{Q_0} |\nabla \phi_n^1|^2 dx.$$

By taking sum, we obtain

$$\int_{\Omega} |\phi_n^1|^p dx \leq C \hat{d}_n^{(1-\theta)(q+1)} \|\phi_n^1\|_{H_0^1(\Omega)}^{\theta(q+1)-2} \int_{\Omega} |\nabla \phi_n^1|^2 dx$$

If  $\hat{d}_n \rightarrow 0$  as  $n \rightarrow \infty$  for some subsequence then  $\|\phi_n^1\|_{L^q(\Omega)} \rightarrow 0$  as  $n \rightarrow \infty$ . On the other hand, by (16),

$$o(1) = I'_\Omega[\phi]\phi = \|\phi_n^1\|_{H_0^1(\Omega)}^2 + \lambda \|\phi_n^1\|_{L^2(\Omega)}^2 - \int_{\Omega} (\phi_n^1)_+^{p+1} dx.$$

By Sobolev's inequality,

$$\int_{\Omega} (\phi_n^1)^{p+1} dx \leq \epsilon C \|\phi_n^1\|_{H_0^1(\Omega)}^2 + C(\epsilon) \|\phi_n^1\|_{L^{q+1}(\Omega)}^{q+1}.$$

So, for sufficiently small  $\epsilon$ , we have

$$\|\phi_n^1\|_{H_0^1(\Omega)}^2 \leq C \|\phi_n^1\|_{L^{q+1}(\Omega)}^{q+1} = o(1) \quad \text{as } n \rightarrow \infty.$$

It is contradiction. So we obtain  $\liminf_{n \rightarrow \infty} \hat{d}_n > 0$ .

Here, by passing to a subsequent if necessary, there is  $j(n) \in \{1, \dots, k\}$  and  $m(n) \in \mathbb{N} \cup \{0\}$  such that  $\|\phi_n^1\|_{Q_{m(n)}^{j(n)}}$ , where  $Q_{m(n)}^j(n) = Q_0$  if  $m(n) = 0$ .

We can assume  $j(n) \equiv j$  by passing to a subsequence if necessary. We show

that  $m(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Suppose that there is a constant  $m_0$  such that  $m(n) \leq m_0$  for all  $n$ . Then

$$d_n^2 \leq \sum_{0 \leq m \leq m_0} \|\phi_n^1\|_{L^2(Q_m^j)}^2 = \|\phi_n^1\|_{L^2(Q)}^2,$$

where  $Q = \bigcup_{0 \leq m \leq m_0} Q_m^j$ . As  $n \rightarrow \infty$ , it contradicts to (15). We can assume that  $m(n)$  is increasing without loss of generality.

Define the map  $\Lambda$  by

$$\Lambda(x) := \Lambda_j(x', x_n + m(n) - 1).$$

Then  $\Lambda(Q_1) = Q_{m(n)}^j$ ,  $\Lambda(\hat{S}_d) = \sum_{m \geq m(n)} Q_m^j$ . Put  $\hat{\phi}_n^1 := \phi_n^1 \circ \Lambda$  then we have

$$\|\hat{\phi}_n^1\|_{H^1(\mathbf{R}^N)} < C, \quad \|\hat{\phi}_n^1\|_{L^2(Q_1)} \geq d_n.$$

By the weak compactness of  $H^1(\mathbf{R}^N)$ , there exists  $\hat{u}^1 \in H^1(\mathbf{R}^N)$  such that

$$\hat{\phi}_n^1 \rightharpoonup \hat{u}^1 \quad \text{weakly in } H^1(\mathbf{R}^N)$$

by passing to a subsequence if necessary. Here, we can assume parallel transformation to  $\Lambda_j$  are  $\Lambda_{j+1}, \dots, \Lambda_{j+\hat{j}}$  for some  $\hat{j} \in \mathbf{N} \cup \{0\}$ . So there is a cone  $V$  such that  $V \cap \Omega \subset V \cap (\Lambda_j(\hat{S}_d) \cup \Lambda_{j+\hat{j}}(\hat{S}_d))$ . It means that for  $n_0 \in \mathbf{N}$ ,

$$\begin{aligned} \hat{\phi}_n^1 &= 0 \text{ on } \Lambda_j^{-1}(V \setminus (\Lambda_j(\hat{S}_d) \cup \dots \cup \Lambda_{j+\hat{j}}(\hat{S}_d)) - (0, m(n_0) - 1)) \\ &= (\Lambda_j^{-1}V - (0, m(n_0) - 1)) \setminus (\hat{S}_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(\hat{S}_d) \dots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(\hat{S}_d)) \\ &\text{if } n \geq n_0. \end{aligned}$$

As  $n \rightarrow \infty$ , we obtain

$$\hat{u}^1 = 0 \text{ on } (\Lambda_j^{-1}V - (0, m(n_0) - 1)) \setminus (\hat{S}_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(\hat{S}_d) \dots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(\hat{S}_d))$$

As  $n_0 \rightarrow \infty$ , we have

$$\hat{u}^1 = 0 \text{ on } \mathbf{R}^N \setminus (\hat{S}_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(\hat{S}_d) \dots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(\hat{S}_d))$$

It means that there is  $\hat{u}^{1,0} \in H_0^1(\hat{S}_d)$ ,  $\hat{u}^{1,1} \in H_0^1(\Lambda_j^{-1} \circ \Lambda_{j+1}(\hat{S}_d))$ ,  $\dots$ ,  $\hat{u}^{1,\hat{j}} \in H_0^1(\Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(\hat{S}_d))$  such that  $\hat{u}^1 = \hat{u}^{1,1} + \dots + \hat{u}^{1,\hat{j}}$ .

Fix any  $\psi \in C_0^\infty(\hat{S}_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(\hat{S}_d) \dots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(\hat{S}_d))$ . Since  $m(n) \rightarrow \infty$  as  $n \rightarrow \infty$ ,  $\Lambda(\text{supp } \psi) \subset \Omega$  for large  $n$ . So we have

$$\begin{aligned} & \left| \int_{\mathbf{R}^N} \nabla \hat{\phi}_n^1 \nabla \psi + \lambda \hat{\phi}_n^1 \psi - (\hat{\phi}_n^1)_+^p \psi \, dx \right| \\ &= \left| \int_{\mathbf{R}^N} \nabla \phi_n^1 \nabla (\psi \circ \Lambda) + \lambda \phi_n^1 (\psi \circ \Lambda) - (\phi_n^1)_+^p \psi \circ \Lambda \, dx \right| \\ &= | \langle I'[\phi_n^1], \psi \circ \Lambda \rangle | \leq o(1) \|\psi \circ \Lambda\|_{H^1(\mathbf{R}^N)} = o(1) \|\psi\|_{H^1(\mathbf{R}^N)}. \end{aligned}$$

As  $n \rightarrow \infty$ , we obtain

$$\int_{\mathbf{R}^N} \nabla \hat{u}^1 \nabla \psi + \lambda \hat{u}^1 \psi - (\hat{u}^1)_+^p \psi \, dx = 0.$$

It means

$$I'[\hat{u}^1] = 0 \quad \text{in } H^{-1}(S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d)).$$

Hence  $\hat{u}^{1,i}$  is a weak solution of

$$\begin{cases} -\Delta \hat{u}^{1,i} + \lambda \hat{u}^{1,i} = (\hat{u}^{1,i})_+^p & \text{in } \Lambda_j^{-1} \circ \Lambda_{j+i}(S_d), \\ \hat{u}^{1,i} \in H_0^1(\Lambda_j^{-1} \circ \Lambda_{j+i}(S_d)) \end{cases}$$

for  $0 \leq i \leq \hat{j}$ . Put  $u^{i+1}(x) := \hat{u}^{1,i} \circ \Lambda_j^{-1}$  and  $z_n^{i+1} := \Lambda_j(x', m(n) - 1)$  with  $\Lambda_j(x', 0) \in \Lambda_{j+i}(\{y' = 0\})$ . for  $0 \leq i \leq \hat{j}$ . Then

$$\begin{cases} -\Delta u^{i+1} + \lambda u^{i+1} = (u^{i+1})_+^p, u^{i+1} > 0 & \text{in } \Lambda_{j+i}(S), \\ u^{i+1} = 0 & \text{on } \partial \Lambda_{j+i}(S), \end{cases}$$

$$\begin{aligned} \phi_n^1(x) &\rightharpoonup u^1(x - z_n^1) + \cdots + u^{1+\hat{j}}(x - z_n^{1+\hat{j}}) && \text{weakly in } H^1(\mathbf{R}^N), \\ \phi_n^1(x) &\rightarrow u^1(x - z_n^1) + \cdots + u^{1+\hat{j}}(x - z_n^{1+\hat{j}}) && \text{in } L_{\text{loc}}^p(\mathbf{R}^N), \\ \phi_n^1(x) &\rightarrow u^1(x - z_n^1) + \cdots + u^{1+\hat{j}}(x - z_n^{1+\hat{j}}) && \text{a.e. in } \mathbf{R}^N \quad \text{as } n \rightarrow \infty \end{aligned}$$

for  $0 \leq i \leq \hat{j}$ . If  $\phi_n^1 \rightarrow u^1(x - z_n^1) + \cdots + u^{1+\hat{j}}(x - z_n^{1+\hat{j}})$  strongly in  $H_0^1(\mathbf{R}^N)$  for some subsequence then the proof is complete.

If not, by using the argument above, inductively, by passing to a subsequence if necessary, we have

$$\begin{aligned} \phi_n^l(x) &= u_n(x) - u^0(x) - u^1(x - z_n^1) - \cdots - u^l(x - z_n^l) + o(1) \text{ weakly in } H_0^1(\mathbf{R}^N), \\ \|\phi_n^l\|_{H_0^1(\mathbf{R}^N)} &= \|u_n\|_{H_0^1(\mathbf{R}^N)} - \|u^0\|_{H_0^1(\mathbf{R}^N)} - \|u^1\|_{H_0^1(\mathbf{R}^N)} - \cdots - \|u^l\|_{H_0^1(\mathbf{R}^N)} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since  $\|u^1\|_{H_0^1(\mathbf{R}^N)}, \dots, \|u^l\|_{H_0^1(\mathbf{R}^N)} \geq c\alpha(S)$  and  $\|u_n\|_{H_0^1(\mathbf{R}^N)}$  is uniformly bounded, there is some  $l \geq 1$  such that  $u_n(x) = u^0(x) + u^1(x - z_n^1) + \cdots + u^l(x - z_n^l) + o(1)$  strongly in  $H_0^1(\mathbf{R}^N)$ . It completes the proof.  $\square$

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